Objective

- Develop and use formulas for the sums of the measures of interior and exterior angles of a polygon.

Why

This human polygonal structure required careful planning and design in order for all of the pieces to fit together properly. How do you think the designers of the figure achieved the final result?

Angle Sums in Polygons

A convex polygon is one in which no part of a line segment connecting any two points on the polygon is outside the polygon. A concave polygon does not have this characteristic. In this book, the word polygon will mean a convex polygon unless otherwise stated.

Activity 1

Sums of Interior Angles

Pentagon ABCDE has been divided into three triangular regions by drawing all possible diagonals from one vertex.

1. Find each of the following:
   \[ m\angle 1 + m\angle 2 + m\angle 3 = \ ? \]
   \[ m\angle 4 + m\angle 5 + m\angle 6 = \ ? \]
   \[ m\angle 7 + m\angle 8 + m\angle 9 = \ ? \]

2. Add the three expressions.
   \[ m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + \cdots + m\angle 9 = \ ? \]
3. Use the diagram and the result from Step 2 to determine the sum of the measures of the interior angles of pentagon ABCDE (that is, \( m \angle EAB + m \angle B + m \angle BCD + m \angle CDE + m \angle E = \) ?).

4. You can form triangular regions by drawing all possible diagonals from a given vertex of any polygon. Complete the table below. Use sketches to illustrate your answers.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of sides</th>
<th>Number of triangular regions</th>
<th>Sum of measures of angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>?</td>
<td>1</td>
<td>180°</td>
</tr>
<tr>
<td>quadrilateral</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>pentagon</td>
<td>?</td>
<td>3</td>
<td>540°</td>
</tr>
<tr>
<td>hexagon</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>n-gon</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

5. Write a formula for the sum of the interior angles of a polygon in terms of the number of sides, \( n \). Complete the formula below.

**Sum of the Interior Angles of a Polygon**

The sum of the measures of the interior angles of a polygon with \( n \) sides is \( ? \).

Recall that a regular polygon is one in which all the angles are congruent and all the sides are congruent. Equilateral triangles and squares are examples of regular polygons. In an equilateral triangle, each angle has a measure of 60°. In a square, each angle has a measure of 90°.

**CHECKPOINT** Complete the chart below. Then complete the formula beneath it.

<table>
<thead>
<tr>
<th>Regular polygon</th>
<th>Number of sides</th>
<th>Sum of measures of interior angles</th>
<th>Measure of one interior angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>?</td>
<td>180°</td>
<td>?</td>
</tr>
<tr>
<td>quadrilateral</td>
<td>?</td>
<td>?</td>
<td>90°</td>
</tr>
<tr>
<td>pentagon</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>hexagon</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>n-gon</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

**The Measure of an Interior Angle of a Regular Polygon**

The measure of an interior angle of a regular polygon with \( n \) sides is \( ? \).
**Activity 2**

**Exterior Angle Sums in Polygons**

1. Draw a triangle and extend each side in one direction to form an exterior angle at each vertex. Find the sum of the measures of the three exterior angles that you formed. Record your results.

2. Cut out the exterior angles and fit them together. Record your results.

3. Repeat Steps 1 and 2 for a quadrilateral.

4. Repeat Steps 1 and 2 for a pentagon.

5. Make a conjecture about the sum of the measures of the exterior angles of a polygon (one at each vertex). You will prove your conjecture in Steps 6–10.

6. What is the sum of the measures of all interior and exterior angles of the triangle at right?

7. What is the sum of the measures of all interior and exterior angles of the quadrilateral at right?

8. Using your results from Steps 6 and 7, write a formula for the sum of the measures of all interior and exterior angles of an \( n \)-gon.

9. Complete the table below.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of sides</th>
<th>Sum of exterior and interior angles</th>
<th>Sum of interior angles</th>
<th>Sum of exterior angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>3</td>
<td>540°</td>
<td>180°</td>
<td>360°</td>
</tr>
<tr>
<td>quadrilateral</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>pentagon</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>hexagon</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( n )-gon</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

10. Use the formula from Activity 1 and the formula from Step 8 above to write an expression for the sum of the measures of the exterior angles of an \( n \)-gon. Use algebra to simplify the expression. Complete the theorem.

**Theorem: Sum of the Exterior Angles of a Polygon**

The sum of the measures of the exterior angles of a polygon is \( 360° \).
Communicate

1. Explain how to tell how many triangular regions can be formed in a polygon by drawing all possible diagonals from one vertex.

2. Is it possible to draw a quadrilateral with three interior angles that measure 60° each? Explain your reasoning.

3. The figures at left can be a “proof without words” of the result you may have discovered in Activity 2. A polygon has its sides extended as rays. Imagine that you are looking at the figure from farther and farther away. Explain why the sum of the exterior angles is 360°.

Guided Skills Practice

4. Find the sum of the measures of the interior angles of an octagon.  
   (Activity 1)

5. Find the sum of the measures of the interior angles of a 13-gon.  
   (Activity 1)

6. Find the sum of the measures of the exterior angles of a heptagon.  
   (Activity 2)

7. Find the sum of the measures of the exterior angles of an 11-gon.  
   (Activity 2)

Practice and Apply

8. Refer to the figure at right to find the indicated measures.
   a. $x = \ ?$
   b. $y = \ ?$
   c. $z = \ ?$

For Exercises 9–14, find the indicated angle measure.

9. $90° \ 75° \ 85°$
10. $132° \ 120° \ ?$
11. $130° \ 100° \ 90° \ 110°$
12. $90° \ ?$
13. $110° \ ? \ 100°$
14. $105° \ ? \ 75°$
For each polygon, determine the measure of an interior angle and the measure of an exterior angle.

15. a rectangle  
16. an equilateral triangle  
17. a regular dodecagon  
18. an equiangular pentagon

For Exercises 19–21, an interior angle measure of a regular polygon is given. Find the number of sides of the polygon.

19. 135°  
20. 150°  
21. 165°

For Exercises 22–24, an exterior angle measure of a regular polygon is given. Find the number of sides of the polygon.

22. 60°  
23. 36°  
24. 24°

For Exercises 25–37, find the indicated angle measure.

25. \( \angle A \)  
26. \( \angle B \)  
27. \( \angle C \)  
28. \( \angle D \)  
29. \( \angle E \)  
30. \( \angle F \)  
31. \( \angle G \)  
32. \( \angle H \)  
33. \( \angle I \)  
34. \( \angle J \)  
35. \( \angle K \)  
36. \( \angle L \)  
37. \( \angle M \)

38. What is the maximum possible number of acute angles in a triangle? Can a triangle have no acute angles? Explain your reasoning.

39. What is the maximum possible number of acute angles in a quadrilateral? Can a quadrilateral have no acute angles? Explain your reasoning.

40. What is the maximum possible number of acute angles in a pentagon? Can a pentagon have no acute angles? Explain your reasoning.

41. Find the sum of the measures of the numbered vertex angles of a 5-pointed star polygon. (Hint: First find the measure of the exterior angle indicated by a question mark in each diagram below.)
42. **GEMOLOGY** Precious stones are often cut in a *brilliant cut* to maximize the amount of light reflected by the stone. The angles of the cut depend on the refractive properties of the type of stone. The optimal angles for a diamond are shown in the cross section below. The cut has reflectional symmetry across the axis shown. Find the measures of the indicated angles in the figure.

![Diamond Cross Section](image)

43. **GEMOLOGY** A brilliant cut topaz should have a pavilion main angle of 40° and a crown angle of 37°. Sketch a cross section of such a gem and find the other angles in the cross section.

**Look Back**

44. How is the distance from a point to a line determined? *(LESSON 1.4)*

45. List all pairs of supplementary angles in the photo below. What are these types of angles called? *(LESSON 1.3)*

**APPLICATION**

46. **TRANSPORTATION** Due to zoning regulations, the measure of an angle at an intersection cannot be less than 75°. If m\(\angle 1 = 75^\circ\), what is m\(\angle 4\)? *(LESSON 1.3)*

47. List all pairs of congruent angles in the photo. What are these types of angles called? *(LESSON 2.2)*

**Look Beyond**

Some regular polygons fit together around a single point with no overlaps or gaps. For example, four squares fit together at a point, as shown at left.

48. What is the measure of each angle at the indicated point? What is the sum of the measures of the angles at this center point?

49. For a regular n-gon to form a pattern like the one described above, the measure of its interior angles must be a factor of 360°. Explain why this is true.

50. What other regular n-gons will fit together around a point? How can you be sure that you have found all of the possible n-gons?

51. **BIOLOGY** A beehive is constructed from regular hexagons, as shown. What do you think are some advantages of using hexagons?